

$$\begin{aligned}
& + \left[C_{02} + C'_2 \left(\frac{s}{m_t^2} \right) - (2I'_3) I^{(2)} \left(\frac{s}{m_t^2} \right) \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \\
& + \left[C_{03} - (2I'_3) I^{(3)} \left(\frac{s}{m_t^2} \right) \right] \left(\frac{\alpha_s(s)}{\pi} \right)^3 + \left[C_{04} - (2I'_3) \delta_{I^{(4)}} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^4 \\
& + \delta_{C05} \left(\frac{\alpha_s(s)}{\pi} \right)^5 + \frac{\overline{m}_c^2(s) + \overline{m}_b^2(s)}{s} C_{23} \left(\frac{\alpha_s(s)}{\pi} \right)^3 + \frac{\overline{m}_q^2(s)}{s} \left[C_{20}^A + C_{21}^A \frac{\alpha_s(s)}{\pi} \right. \\
& \left. + C_{22}^A \left(\frac{\alpha_s(s)}{\pi} \right)^2 + 6 \left(3 + \ln \frac{m_t^2}{s} \right) \left(\frac{\alpha_s(s)}{\pi} \right)^2 + C_{23}^A \left(\frac{\alpha_s(s)}{\pi} \right)^3 \right] \\
& - 10 \frac{\overline{m}_q^4(s)}{m_t^2} \left[\frac{8}{81} + \frac{1}{54} \ln \frac{m_t^2}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \\
& + \frac{\overline{m}_c^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_c^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 + \frac{\overline{m}_b^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_b^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \\
& + \frac{\overline{m}_q^4(s)}{s^2} \left\{ C_{40}^A + C_{41}^A \frac{\alpha_s(s)}{\pi} + \left[C_{42}^A + C_{42}^{AL} \ln \frac{\overline{m}_q^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \right. \\
& \left. - 12 \frac{\overline{m}_q^4(s)}{s^2} \left(\frac{\alpha_s(s)}{\pi} \right)^2 \right\} \tag{B.2}
\end{aligned}$$

In Eq. (B.1) and Eq. (B.2) only the charm and bottom quark finite mass corrections are retained, i.e. $\overline{m}_q = 0$ for $q = u, d, s$. As a consequence, these corrections are only valid above the bottom threshold (≈ 10 GeV) and below the top threshold (≈ 340 GeV). The mass \overline{m}'_q denotes the other quark mass, i.e., it is \overline{m}_b if $q = c$ and \overline{m}_c if $q = b$. The running of the quark masses are computed in the $\overline{\text{MS}}$ scheme according to Eq. (3.16). The numerical coefficients in Eq. (B.1) and Eq. (B.2) are listed below:

$$\begin{aligned}
R_q^d(s) = & 1 + \frac{3}{4} Q_q^2 \frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{1}{4} Q_q^2 \frac{\alpha(s)}{\pi} \frac{\alpha_s(s)}{\pi} \\
& + \left[C_{02} + C'_2 \left(\frac{s}{m_t^2} \right) \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 + C_{03} \left(\frac{\alpha_s(s)}{\pi} \right)^3 + C_{04} \left(\frac{\alpha_s(s)}{\pi} \right)^4 \\
& + \delta_{C05} \left(\frac{\alpha_s(s)}{\pi} \right)^5 + \frac{\overline{m}_c^2(s) + \overline{m}_b^2(s)}{s} C_{23} \left(\frac{\alpha_s(s)}{\pi} \right)^3 \\
& + \frac{\overline{m}_q^2(s)}{s} \left[C_{21}^V \frac{\alpha_s(s)}{\pi} + C_{22}^V \left(\frac{\alpha_s(s)}{\pi} \right)^2 + C_{23}^V \left(\frac{\alpha_s(s)}{\pi} \right)^3 \right] \\
& + \frac{\overline{m}_c^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_c^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 + \frac{\overline{m}_b^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_b^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \\
& + \frac{\overline{m}_q^4(s)}{s^2} \left\{ C_{41}^V \frac{\alpha_s(s)}{\pi} + \left[C_{42}^V + C_{42}^{VL} \ln \frac{\overline{m}_q^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \right\} \\
& + 12 \frac{\overline{m}_q^{t4}(s)}{s^2} \left(\frac{\alpha_s(s)}{\pi} \right)^2 - \frac{\overline{m}_q^6(s)}{s^3} \left\{ 8 + \frac{16}{27} \left[155 + 6 \ln \frac{\overline{m}_q^2(s)}{s} \right] \frac{\alpha_s(s)}{\pi} \right\} \tag{B.3} \\
R_q^d(s) = & 1 + \frac{3}{4} Q_q^2 \frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{1}{4} Q_q^2 \frac{\alpha(s)}{\pi} \frac{\alpha_s(s)}{\pi} \tag{B.4} \\
& + \left[\frac{151}{162} - \frac{1}{18} \zeta(2) - \frac{19}{27} \zeta(3) \right] n_f; \tag{B.5}
\end{aligned}$$

APPENDIX B

Radiator Functions

The radiator functions describe the final state QED and QCD vector (Eq. (B.1)) and axial vector (Eq. (B.2)) corrections of hadronic Z decays. In addition, they contain mixed QED \otimes QCD corrections and finite mass corrections (the latter are accounted for in terms of running masses).

The following radiator functions are taken from [27] and are updated by the new NNNLO calculation from [42].

$$\begin{aligned}
R_q^d(s) = & 1 + \frac{3}{4} Q_q^2 \frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{1}{4} Q_q^2 \frac{\alpha(s)}{\pi} \frac{\alpha_s(s)}{\pi} \\
& + \left[C_{02} + C'_2 \left(\frac{s}{m_t^2} \right) \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 + C_{03} \left(\frac{\alpha_s(s)}{\pi} \right)^3 + C_{04} \left(\frac{\alpha_s(s)}{\pi} \right)^4 \\
& + \delta_{C05} \left(\frac{\alpha_s(s)}{\pi} \right)^5 + \frac{\overline{m}_c^2(s) + \overline{m}_b^2(s)}{s} C_{23} \left(\frac{\alpha_s(s)}{\pi} \right)^3 \\
& + \frac{\overline{m}_q^2(s)}{s} \left[C_{21}^V \frac{\alpha_s(s)}{\pi} + C_{22}^V \left(\frac{\alpha_s(s)}{\pi} \right)^2 + C_{23}^V \left(\frac{\alpha_s(s)}{\pi} \right)^3 \right] \\
& + \frac{\overline{m}_c^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_c^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 + \frac{\overline{m}_b^4(s)}{s^2} \left[C_{42} - \ln \frac{\overline{m}_b^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \\
& + \frac{\overline{m}_q^4(s)}{s^2} \left\{ C_{41}^V \frac{\alpha_s(s)}{\pi} + \left[C_{42}^V + C_{42}^{VL} \ln \frac{\overline{m}_q^2(s)}{s} \right] \left(\frac{\alpha_s(s)}{\pi} \right)^2 \right\} \tag{B.3} \\
& + 12 \frac{\overline{m}_q^{t4}(s)}{s^2} \left(\frac{\alpha_s(s)}{\pi} \right)^2 - \frac{\overline{m}_q^6(s)}{s^3} \left\{ 8 + \frac{16}{27} \left[155 + 6 \ln \frac{\overline{m}_q^2(s)}{s} \right] \frac{\alpha_s(s)}{\pi} \right\} \tag{B.4} \\
R_q^d(s) = & 1 + \frac{3}{4} Q_q^2 \frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{1}{4} Q_q^2 \frac{\alpha(s)}{\pi} \frac{\alpha_s(s)}{\pi} \tag{B.5}
\end{aligned}$$

$$C_{04} = -156.61 + 18.77n_f - 0.7974n_f^2 + 0.0215n_f^3; \quad (\text{B.6})$$

Quadratic massive corrections [72]:

$$C_{23} = -80 + 60\zeta(3) + \left[\frac{32}{9} - \frac{8}{3}\zeta(3) \right] n_f, \quad (\text{B.7})$$

$$C_{21}^V = 12, \quad (\text{B.8})$$

$$C_{22}^V = \frac{253}{2} - \frac{13}{3}n_f, \quad (\text{B.9})$$

$$\begin{aligned} C_{23}^V &= 2522 - \frac{855}{2}\zeta(2) + \frac{310}{3}\zeta(3) - \frac{5225}{6}\zeta(5) \\ &\quad + \left[-\frac{4942}{27} + 34\zeta(2) - \frac{394}{27}\zeta(3) + \frac{1045}{27}\zeta(5) \right] n_f + \left[\frac{125}{54} - \frac{2}{3}\zeta(2) \right] n_f^2, \end{aligned} \quad (\text{B.10})$$

$$C_{20}^A = -6, \quad (\text{B.11})$$

$$C_{21}^A = -22, \quad (\text{B.12})$$

$$\begin{aligned} C_{22}^A &= -\frac{8221}{24} + 57\zeta(2) + 117\zeta(3) + \left[\frac{151}{12} - 2\zeta(2) - 4\zeta(3) \right] n_f, \\ C_{23}^A &= -\frac{4544045}{864} + 1340\zeta(2) + \frac{118915}{36}\zeta(3) - 127\zeta(5) \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} &\quad + \left[\frac{71621}{162} - \frac{209}{2}\zeta(2) - 216\zeta(3) + 5\zeta(4) + 55\zeta(5) \right] n_f \\ &\quad + \left[-\frac{13171}{1944} + \frac{16}{9}\zeta(2) + \frac{26}{9}\zeta(3) \right] n_f^2; \end{aligned} \quad (\text{B.14})$$

Quartic massive corrections [72]:

$$C_{42} = \frac{13}{3} - 4\zeta(3), \quad (\text{B.15})$$

$$C_{40}^V = -6, \quad (\text{B.16})$$

$$C_{41}^V = -22, \quad (\text{B.17})$$

$$\begin{aligned} C_{42}^V &= -\frac{3029}{12} + 162\zeta(2) + 112\zeta(3) + \left[\frac{143}{18} - 4\zeta(2) - \frac{8}{3}\zeta(3) \right] n_f, \\ C_{42}^{VL} &= -\frac{11}{2} + \frac{1}{3}n_f, \end{aligned} \quad (\text{B.18}) \quad (\text{B.19})$$

$$C_{40}^A = 6, \quad (\text{B.20})$$

$$C_{41}^A = 10, \quad (\text{B.21})$$

$$C_{42}^A = \frac{3389}{12} - 162\zeta(2) - 220\zeta(3) + \left[-\frac{41}{6} + 4\zeta(2) + \frac{16}{3}\zeta(3) \right] n_f, \quad (\text{B.22})$$

$$C_{42}^{AL} = \frac{77}{2} - \frac{7}{3}n_f; \quad (\text{B.23})$$

Power suppressed top-mass correction [72]:

$$C_2^L(x) = x \left(\frac{44}{675} - \frac{2}{135} \ln x \right); \quad (\text{B.24})$$

Singlet axial corrections:

$$I^{(2)}(x) = -\frac{37}{12} + \ln x + \frac{7}{81}x + 0.0132x^2, \quad (\text{B.25})$$

$$I^{(3)}(x) = -\frac{5075}{216} + \frac{23}{6}\zeta(2) + \zeta(3) + \frac{67}{18}\ln x + \frac{23}{12}\ln^2 x; \quad (\text{B.26})$$

Singlet vector correction [72]:

$$R_0^L(s) = \left(\sum_f v_f \right)^2 \left(-0.41317 \right) \left(\frac{\alpha_s(s)}{\pi} \right)^3; \quad (\text{B.27})$$

$$\delta_{I^{(4)}} = \frac{I^{(3)}(x)}{I^{(2)}(x)} I^{(3)}(x) (1 - \Delta_{I^{(4)}}) \quad (\text{B.28})$$

$$\delta_{C05} = -68.078(1 - \Delta_{C05}) \quad (\text{B.29})$$

The prefactors are rough estimates from perturbative QCD series. They simulate the seize of the unknown higher orders. The parameters $\Delta_{I^{(3)}}$ and Δ_{C05} are allowed to vary between zero and two. These uncertainties are only taken into account for the determination of the theoretical uncertainty of $\alpha_S(M_Z^2)$. By default $\Delta_{I^{(3)}}$ and Δ_{C05} are fixed to one.